MATH2050C Assignment 8

Deadline: March 14 , 2018.

Hand in: 4.1 no. 11b, 12d, 15; 4.2 no. 1c, 11d, 12. Supp; Ex. no. 3.

Section 4.1 no. 7, 8, 9bd, 10b, 11b, 12bd, 15.

Section 4.2 no. 1, bc, 11 cd, 12.

Supplementary Exercises

1. Let f be function defined on (a, b) except possibly at $x_0 \in (a, b)$. It is has a **right hand limit** at x_0 if there exists some L such that for all $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in (x_0, x_0 + \delta) \cap (a, b)$. Denote it by $L = \lim_{x \to x_0^+} f(x)$. Similarly we define the **left hand limit** of f at x_0 and denote it by $\lim_{x \to x_0^-} f(x)$. Show that $\lim_{x \to x_0} f(x)$ exists if and only if both one-sided limits exist and are equal.

2. Let f be defined on (a, b) possibly except $x_0 \in (a, b)$. Show that $\lim_{x \to x_0} |f(x)| = |L|$ whenever $\lim_{x \to x_0} f(x) = L$.

3. Let f be defined on (a, b) possibly except $x_0 \in (a, b)$. Suppose that $\lim_{x \to x_0} f(x) = L$ for some L. Show that $\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$ provided $f \geq 0$ on (a, b). Suggestion: Consider $L > 0$ and $L = 0$ separately.

Comments on Limits of Functions

Let $x_0 \in (a,b)$ and f a function defined on (a,b), possibly except at x_0. In this chapter we only consider two kinds of limits, first $\lim_{x\to x_0} f(x)$ and second $\lim_{x\to x_0^+} f(x)$ or $\lim_{x\to x_0^-} f(x)$ (one-sided limit). In Text the limit at a cluster point is discussed. However, these two special cases suffice for all later development.

There is a localization principle hidden behind the discussion. Let us single it out. The proof follows easily from the definition of the limit of functions.

Proposition 8.1 (Localization Principle) Let $x_0 \in (a,b)$ and f a function defined on (a,b), possibly except at x_0. Let $(c,d) \setminus \{x_0\}$ be another interval on which f is also defined. The existence and limit of f regarded as a function on $(a,b) \setminus \{x_0\}$ are the same as the existence and limit of f regarded as a function on $(c,d) \setminus \{x_0\}$.

The Squeeze Theorem is very often used in the following form.

Theorem 8.2 (Squeeze Theorem) Let f and g defined on $(a,b) \setminus \{x_0\}$. Suppose that there exists an interval $(x_0 - \delta_0, x_0 + \delta_0) \subset (a,b)$ such that

$$|f(x) - L| \le g(x) \ , \quad \forall x, \ 0 < |x - x_0| < \delta_0 \ .$$

Then $\lim_{x\to x_0} f(x) = L$ if $\lim_{x\to x_0} g(x) = 0$.

Proof. It suffices to look at the equivalent form

$$-g(x) \le f(x) - L \le g(x) \ ,$$

and apply the localization principle to $(x_0 - \delta, x_0 + \delta)$ instead of (a,b).

We note the following result.

Theorem 8.3. Let f be defined on (a,b) possibly except $x_0 \in (a,b)$. Suppose there exists some L such that for every $\varepsilon > 0$, there is some δ such that

$$|f(x) - L| < M\varepsilon \ , \quad \forall x, \ 0 < |x - x_0| < \delta \ ,$$

where M is a constant independent of ε. Then $\lim_{x\to x_0} f(x) = L$.

Proof. Let $\varepsilon > 0$ be given. For $\varepsilon' = \varepsilon/M$ which is again a positive number, there is some δ such that $|f(x) - L| < M\varepsilon' = \varepsilon$ for all $x, \ 0 < |x - x_0| < \delta$, so $\lim_{x\to x_0} f(x) = L$.

Finally, we summarize ways to establish convergence, that is, the existence of limits of functions:

- Use ε-δ definition,
- Use Limit Theorem,
- Use Squeeze Theorem.

There are two ways to establish divergence, that is, the non-existence of limits:

- $f(x)$ becomes unbounded near x_0,
- There exist two sequences tending to x_0 with different limits.